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THE THEORETICAL BEHAVIOR OF A POLYCRYSTALLINE SOLID AS RELATED TO CERTAIN GENERAL CONCEPTS OF CONTINUUM PLASTICITY

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Abstract—This paper is directed toward establishing general characteristics of continuum stress–plastic strain relations from Schmid's Law of plastic slip in individual crystal grains. To this end certain theoretical results obtained by Lin are reaffirmed through a rigorous derivation, and the principle of maximum plastic work is extended to small elastic–plastic strains in an isotropic polycrystalline aggregate. It is shown that the macroscopic incremental plastic strain vector over a unit volume of a fine-grained solid is strictly normal to a yield surface in macrostress space only if the individual crystals are elastically isotropic. The resulting equations for polycrystalline solids are contrasted with those obtained from certain stability postulates and thermodynamic foundations in continuum plasticity, and general features of similarity are discussed.

1. INTRODUCTION

IN parallel with the development of the mathematical theory of plasticity of strain-hardening solids, several investigators have pursued what might be termed a physical theory of plasticity—that is, the prediction of the behavior of a polycrystalline metal based on the experimentally determined stress–strain law of individual crystal grains. Taylor, after early experimental work with his associates on single crystals of aluminum [1–4], established analytically a close upper bound on the stress–strain curve of a tensile specimen [5, 6] by neglecting the elastic strains and assuming a uniform plastic strain field throughout the aggregate. Taylor's analytical results were rederived by Bishop and Hill [7, 8] and further discussed by Bishop [9] and Taylor [10]. This assumption of homogeneous plastic strain corresponds to a kinematically admissible field in an aggregate of rigid–plastic crystals but not to a statically admissible one, since all the equilibrium conditions between grains are not satisfied. Lin [11] extended Taylor's analysis by considering the elastic strains of the individual crystals and assuming uniform total strain throughout, a procedure which also leads to an upper bound. Budiansky, Hashin, and Sanders [12] relaxed the condition of uniform aggregate strain and accounted for equilibrium of a slipped crystal surrounded by grains that are still elastic by using an inclusion analysis given by Eshelby [13]. Interaction between slipped crystals was treated in an approximate manner by Kröner [14] and by Budiansky and Wu [15]; additional calculations were made by Hutchinson [16, 17]. Quantitative evaluations of aggregate behavior that satisfy all equilibrium and continuity conditions have been given for isotropic crystals by Lin and his associates [18 through 25].

In addition to the studies directed toward obtaining quantitative results for polycrystalline aggregates, several significant investigations have been devoted to establishing the general characteristics of continuum stress–plastic strain relations based upon the

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same starting point (Schmid's Law for plastic slip in the individual crystal grains). Bishop and Hill [7] established a principle of maximum plastic work for a slipped crystal and extended this to a rigid–plastic polycrystalline aggregate, obtaining a relation identical to one of the consequences of Drucker's stability postulate [26, 27]. Lin [28] discussed the significance of the elastic properties of the crystals in the overall aggregate behavior and obtained several important results related to the theoretical concepts of normality and the plastic potential. Subsequent theoretical papers concerned with the evolution of continuum (or macro) behavior from crystalline (or micro) behavior include those of Mandel [29], Hill [30–32], and Axelrad and Yong [33].

The present paper rigorously rederives and reaffirms the theoretical conclusions of Lin [28] and extends Bishop and Hill's principle of maximum plastic work [7] to an aggregate of elastically isotropic crystals, thereby proving convexity of the ma roscopic yield surface for small elastic–plastic strains. In addition, the equations for polycrystalline aggregates are related to and compared with certain results from the continuum theories of plasticity.

2. MICROSCOPIC STRESS–STRAIN RELATIONS IN THE CRYSTAL

Consider a unit cube subjected to a macroscopic state of stress σ_{ij} applied as uniform surface traction (Fig. 1). This element represents the smallest differential volume stressed

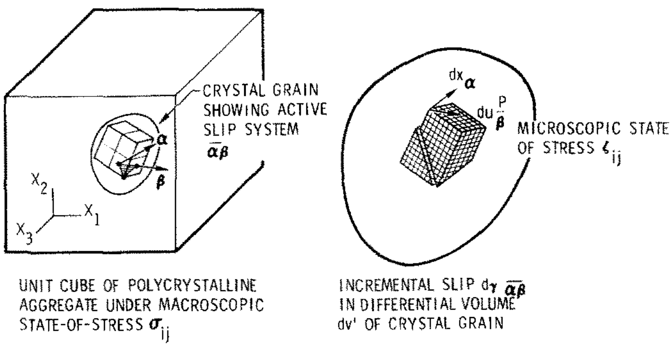

FIG. 1. Plastic slip in crystal grain of polycrystalline aggregate.

in a continuum sense in a finite body subjected to the uniform stress state σ_{ij}. The unit cube itself is inhomogeneous and contains a large number of crystal grains that have various orientations. The (generally) inhomogeneous microscopic stress state within the cube is denoted as ζ_{ij}. The elastic and plastic components of the differential macroscopic small-strain tensor ϵ_{ij} are defined in terms of the components of the differential microscopic strain tensor ξ_{ij} as

$$\mathrm{d}\epsilon^e_{ij} = \int_{V'} (\mathrm{d}\xi^e_{ij})\,\mathrm{d}V', \qquad \mathrm{d}\epsilon^p_{ij} = \int_{V'} (\mathrm{d}\xi^p_{ij})\,\mathrm{d}V', \tag{1}$$

where the integration is over the unit cube. The macroscopic deformation is required to be uniform throughout the assumed homogeneous continuum. Hence, the distribution and orientation of crystals (and the microscopic strain distribution) are taken to be identical in all cubes.

The incremental strain–stress relation within an individual crystal grain having n slip systems is

$$\mathrm{d}\xi_{ij} = c_{ijkl}\,\mathrm{d}\zeta_{kl} + \sum_{\overline{\alpha\beta}} \mathrm{d}\xi_{ij}^{(\overline{\alpha\beta})}. \tag{2}$$

The c_{ijkl} are the elastic compliances of the crystal (which may be anisotropic) and $\mathrm{d}\xi_{ij}^{(\overline{\alpha\beta})}$ is the increment in plastic strain caused by an incremental slip $\mathrm{d}\gamma_{\overline{\alpha\beta}}$ in slip system $\overline{\alpha\beta}$ (α denoting the normal to the slip plane and β the direction of slip). The summation is taken over all the n slip systems of the crystal, several of which, of course, may not be active. For a single slip in slip system $\overline{\alpha\beta}$, there are only two nonzero incremental plastic-strain components in the Cartesian coordinate system that coincides with the directions α and β:

$$\mathrm{d}\xi_{\alpha\beta}^{p} = \mathrm{d}\xi_{\beta\alpha}^{p} = \frac{1}{2}\,\mathrm{d}\gamma_{\overline{\alpha\beta}} = \frac{1}{2}\frac{\partial}{\partial x_\alpha}(\mathrm{d}u_\beta^p) \tag{3}$$

where $\mathrm{d}\gamma_{\overline{\alpha\beta}}$ is the incremental plastic shear detrusion and $\mathrm{d}u_\beta^p$ is the incremental microscopic displacement in direction β due to slip in the $\overline{\alpha\beta}$ system. Thus, from tensor transformation laws

$$\mathrm{d}\xi_{ij}^{(\overline{\alpha\beta})} = \tfrac{1}{2}(n_{\alpha i}n_{\beta j} + n_{\beta i}n_{\alpha j})\,\mathrm{d}\gamma_{\overline{\alpha\beta}} \quad \text{(no summation)}. \tag{4}$$

The $n_{\alpha i}$ are the direction cosines between the α, β coordinate system in the crystal and the x_i coordinate system of the unit cube. Superimposing the contributions of the various slip systems in the case of multiple slip,

$$\mathrm{d}\xi_{ij}^{p} = \sum_{\overline{\alpha\beta}} \mathrm{d}\xi_{ij}^{(\overline{\alpha\beta})} = \tfrac{1}{2}\sum_{\overline{\alpha\beta}} (n_{\alpha i}n_{\beta j} + n_{\beta i}n_{\alpha j})\,\mathrm{d}\gamma_{\overline{\alpha\beta}}. \tag{5}$$

Similarly, the total incremental plastic strain in a slip system $\overline{\delta\eta}$ is (with summation on repeated indices i, j)

$$\mathrm{d}\gamma_{\delta\eta}^{p} = 2n_{\delta i}n_{\eta j}\,\mathrm{d}\xi_{ij}^{p} = \sum_{\overline{\alpha\beta}} (n_{\delta i}n_{\alpha i}n_{\eta j}n_{\beta j} + n_{\delta i}n_{\beta i}n_{\eta j}n_{\alpha j})\,\mathrm{d}\gamma_{\overline{\alpha\beta}} \tag{6}$$

which can be written

$$\mathrm{d}\gamma_{\delta\eta}^{p} = \sum_{\overline{\alpha\beta}} \{\cos(\delta, \alpha)\cos(\eta, \beta) + \cos(\delta, \beta)\cos(\eta, \alpha)\}\,\mathrm{d}\gamma_{\overline{\alpha\beta}}. \tag{7}$$

Note that $\mathrm{d}\gamma_{\overline{\delta\eta}}$ would be that part of $\mathrm{d}\gamma_{\delta\eta}^{p}$ caused by slip in the $\overline{\delta\eta}$ slip system alone. Since slip systems are not generally orthogonal, slip in other systems contributes to the resulting incremental plastic strain in the $\overline{\delta\eta}$ system.

Consider now the resolution of the stress tensor within the crystal. Denoting by $\tau_{\alpha\beta}$ the shear stress in the $\overline{\alpha\beta}$ slip plane,

$$\tau_{\alpha\beta} = n_{\alpha i}n_{\beta j}\zeta_{ij} \tag{8}$$

where ζ_{ij} may vary with position within an individual grain as well as from grain to grain within the unit cube. Taking $\tau_{\alpha\beta}$ to be equal to the critical shear stress $\tau_{\overline{\alpha\beta}}^{(c)}$ necessary to produce plastic slip in the $\overline{\alpha\beta}$ plane, then the incremental mechanical energy-dissipation density during multiple slip can be expressed in terms of this stress. Thus, from equations

(5), (6), (8), and the symmetry of the stress tensor,

$$dw^p \equiv \zeta_{ij}\,d\xi^p_{ij} = \sum_{\overline{\alpha\beta}} \tau^{(c)}_{\overline{\alpha\beta}}\,d\gamma_{\overline{\alpha\beta}}. \tag{9}$$

Defining another microstress state ζ^*_{ij} (for which the corresponding shear stresses $\tau^*_{\alpha\beta}$ on the various slip planes are everywhere less than the critical stress) and proceeding as in Bishop and Hill [7], we obtain the principle of maximum plastic work within an individual crystal grain:

$$(\zeta_{ij}-\zeta^*_{ij})\,d\xi^p_{ij} = \sum_{\overline{\alpha\beta}} (\tau^{(c)}_{\overline{\alpha\beta}}-\tau^*_{\alpha\beta})\,d\gamma_{\overline{\alpha\beta}} \geq 0. \tag{10}$$

Bishop and Hill used this principle to deduce normality and convexity conditions for the yield surface of the crystal in microstress space. Alternately, Lin [28] establishes normality from a comparison of equations (4) and (8). If we denote by $f_{\overline{\alpha\beta}}$ the hyperplane in ζ_{ij} stress space corresponding to the $\overline{\alpha\beta}$ slip system, then

$$f_{\overline{\alpha\beta}} \equiv n_{\alpha i}n_{\beta j}\zeta_{ij} - \tau^{(c)}_{\overline{\alpha\beta}} = 0. \tag{11}$$

In terms of the six-dimensional linear vector space spanned by ζ_{11}, ζ_{12}, ζ_{13}, ζ_{22}, ζ_{23}, and ζ_{33}, the vector gradient $\mathbf{V}_\zeta$ of $f_{\overline{\alpha\beta}}$ has components

$$\mathbf{V}_\zeta f_{\overline{\alpha\beta}} = (n_{\alpha 1}n_{\beta 1},\ n_{\alpha 1}n_{\beta 2}+n_{\beta 1}n_{\alpha 2},\ n_{\alpha 1}n_{\beta 3}+n_{\beta 1}n_{\alpha 3},\ n_{\alpha 2}n_{\beta 2},\ n_{\alpha 2}n_{\beta 3}+n_{\beta 2}n_{\alpha 3},\ n_{\alpha 3}n_{\beta 3}). \tag{12}$$

In single slip, the six incremental "physical" plastic strain components $d\xi^p_{11}$, $2d\xi^p_{12}$, $2d\xi^p_{13}$, $d\xi^p_{22}$, $2d\xi^p_{23}$, $d\xi^p_{33}$, are, from equation (4),

$$d\boldsymbol{\xi}^p = (n_{\alpha 1}n_{\beta 1},\ n_{\alpha 1}n_{\beta 2}+n_{\beta 1}n_{\alpha 2},\ n_{\alpha 1}n_{\beta 3}+n_{\beta 1}n_{\alpha 3},\ n_{\alpha 2}n_{\beta 2},\ n_{\alpha 2}n_{\beta 3}+n_{\beta 2}n_{\alpha 3},\ n_{\alpha 3}n_{\beta 3})\,d\gamma^p_{\alpha\beta}. \tag{13}$$

Thus, the incremental plastic strain vector $d\boldsymbol{\xi}^p$ in the crystal is seen to be normal to the yield surface in six-dimensional microstress space. The yield surface obviously is convex.

3. MACROSCOPIC STRESS–STRAIN RELATIONS IN THE AGGREGATE

First consider a fine-grained aggregate of elastically isotropic crystals. Before the initiation of slip in any crystal of the unit cube, the microstress and microstrain fields are homogeneous and equal to the corresponding fields at the continuum level:

$$\zeta_{ij}(x) = \sigma_{ij}, \qquad \epsilon_{ij} = \xi_{ij}(x) = \epsilon^e_{ij} \tag{14}$$

and

$$\tau_{\alpha\beta}(x) = n_{\alpha i}n_{\beta j}\sigma_{ij} \tag{15}$$

where (x) indicates an arbitrary point within the unit cube. Thus, the initial yield surface in macrostress space is the inner bound of the yield surfaces of the individual crystals. The hyperplane $f_{\overline{\alpha\beta}}$, corresponding to the first slip system, is tangent to the yield surface of the aggregate, and the incremental plastic strain vector $d\boldsymbol{\epsilon}^p$ is normal to this plane. After plastic straining has taken place, the microstress field ζ_{ij} is no longer uniform:

$$\zeta_{ij}(x) = \sigma_{ij} + \zeta^{(s)}_{ij} \tag{16}$$

where the superscript (s) denotes that part of the stress caused by previous crystal slips.

In an active slip system $\overline{\alpha\beta}$, the critical shear stress is

$$\tau^{(c)}_{\overline{\alpha\beta}} = \tau_{\alpha\beta}(x) \equiv n_{\alpha i} n_{\beta j} \sigma_{ij} + \tau^{(R)}_{\alpha\beta}. \tag{17}$$

The first term on the right-hand side of this equation is the active resolved shear stress on the $\overline{\alpha\beta}$ plane (from the continuing elastic isotropy of the crystals); the second term is the residual shear stress resulting from previous plastic slips. By using equations (11), (16), and (17), the yield hyperplane of the $\overline{\alpha\beta}$ slip system can be expressed in terms of the macro-stress state as

$$f_{\overline{\alpha\beta}} = n_{\alpha i} n_{\beta j} \sigma_{ij} - (\tau^{(c)}_{\overline{\alpha\beta}} - \tau^{(R)}_{\alpha\beta}) = 0. \tag{18}$$

From equation (15), the initial yield surface of the isotropic crystal can be expressed

$$f^0_{\overline{\alpha\beta}} = n_{\alpha i} n_{\beta j} \sigma^0_{ij} - \tau^{(c)0}_{\overline{\alpha\beta}} = 0. \tag{19}$$

We see, by comparing these equations, that, even if the crystal is assumed to harden isotropically in microstress space (such that $\tau^{(c)}_{\overline{\alpha\beta}}$ increases equally in both active and passive slip systems, as proposed by Taylor [5]), the macroscopic yield surface cannot simply expand but must translate and perhaps distort as well. This is caused by the presence of $\tau^{(R)}_{\alpha\beta}$ in equation (18), and by its variation from plane to plane, as well as by the variation of both $\tau^{(c)}_{\overline{\alpha\beta}}$ and $\tau^{(R)}_{\alpha\beta}$ from one crystal to another. For isotropic hardening of the crystal, the critical shear stress is taken to be a function of the sum of slips integrated over the deformation path:

$$\tau^{(c)}_{\overline{\alpha\beta}} = \tau^{(c)}_{\overline{\alpha\beta}}\left(\int \sum_{\overline{\alpha\beta}} \mathrm{d}\gamma_{\overline{\alpha\beta}}\right) \tag{20}$$

with $\tau^{(c)}_{\overline{\alpha\beta}}$ equal to $\tau^{(c)0}_{\overline{\alpha\beta}}$ before the initiation of plastic strain in the crystal. The residual shear stress $\tau^{(R)}_{\alpha\beta}$ on the plane of impending slipping is a function of the history of plastic deformation throughout all the crystals of the unit cube.

Consider now the question of normality. The gradient of the yield hyperplane $f_{\overline{\alpha\beta}}$ in the six-dimensional macroscopic-stress space has the components (equation 18).

$$\nabla f_{\overline{\alpha\beta}} = (n_{\alpha 1} n_{\beta 1},\ n_{\alpha 1} n_{\beta 2} + n_{\beta 1} n_{\alpha 2},\ n_{\alpha 1} n_{\beta 3} + n_{\beta 1} n_{\alpha 3},\ n_{\alpha 2} n_{\beta 2},\ n_{\alpha 2} n_{\beta 3} + n_{\beta 2} n_{\alpha 3},\ n_{\alpha 3} n_{\beta 3}) \tag{21}$$

which are identical with equation (12). In single slip, the incremental macroscopic plastic strain is, from equation (1),

$$\mathrm{d}\epsilon^p_{ij} = \mathrm{d}\xi^p_{ij}\,\Delta V' \tag{22}$$

where $\Delta V'$ is that part of the crystal volume experiencing the incremental slip. Thus, from equation (4), since in single slip $\mathrm{d}\xi^p_{ij} \equiv \mathrm{d}\xi^{(\overline{\alpha\beta})}_{ij}$,

$$\mathrm{d}\epsilon^p_{ij} = \tfrac{1}{2}(n_{\alpha i} n_{\beta j} + n_{\beta i} n_{\alpha j})\,\mathrm{d}\gamma_{\overline{\alpha\beta}}\,\Delta V', \tag{23}$$

and the six-dimensional, incremental, macroscopic plastic strain vector is (using equation 21)

$$\mathrm{d}\boldsymbol{\epsilon}^p = \nabla f_{\overline{\alpha\beta}}\,(\mathrm{d}\gamma_{\overline{\alpha\beta}}\,\Delta V'). \tag{24}$$

Hence, this vector $\mathrm{d}\boldsymbol{\epsilon}^p$, representing plastic strain in the aggregate but corresponding to microscopic slip in a single crystal slip system $\overline{\alpha\beta}$, is normal to the yield surface (yield hyperplane) in macroscopic stress space.

For multiple slip in a single crystal, denoting the above plastic strain by $\mathrm{d}\epsilon^{p(\overline{\alpha\beta})}$,

$$\mathrm{d}\boldsymbol{\epsilon}^{p} = \sum_{\overline{\alpha\beta}} \mathrm{d}\boldsymbol{\epsilon}^{p(\overline{\alpha\beta})} = \sum_{\overline{\alpha\beta}} \nabla f_{\overline{\alpha\beta}}\,(\mathrm{d}\gamma_{\overline{\alpha\beta}}\,\Delta V'). \tag{25}$$

For simultaneous multiple slip in two or more crystals,

$$\mathrm{d}\epsilon^{p}_{ij} = \sum_{k} \mathrm{d}\xi^{p}_{ij(k)}\Delta V'_{(k)} = \tfrac{1}{2}\sum_{k}\sum_{\overline{\alpha\beta}} (n_{\alpha i}n_{\beta j} + n_{\beta i}n_{\alpha j})\,\mathrm{d}\gamma_{\overline{\alpha\beta}(k)}\Delta V'_{(k)} \tag{26}$$

where the subscript (k) indicates a crystal volume over which slip takes place and $\sum_k \Delta V'_{(k)} < 1$. Thus, the resultant incremental macroscopic plastic strain vector can be expressed

$$\mathrm{d}\boldsymbol{\epsilon}^{p} = \sum_{k}\sum_{\overline{\alpha\beta}} \mathrm{d}\boldsymbol{\epsilon}^{p(\overline{\alpha\beta})}_{(k)} = \sum_{k}\sum_{\overline{\alpha\beta}} \nabla f_{\overline{\alpha\beta}(k)}(\mathrm{d}\gamma_{\overline{\alpha\beta}(k)}\Delta V'_{(k)}). \tag{27}$$

The yield hyperplanes in macrostress space, corresponding to the active slip systems of those crystals experiencing simultaneous slip, locally bound the domain of elastic response of the unit cube and intersect at the single stress point $(\sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33})$. Since the multipliers of the vector gradients of the $f_{\overline{\alpha\beta}(k)}$ are all positive, the resultant incremental plastic strain vector lies within the pyramid of normals to these intersecting hyperplanes (Fig. 2). Thus, from equations (24) and (27), $\mathrm{d}\boldsymbol{\epsilon}^{p}$ is normal (or within the pyramid of normals) to the subsequent yield surface of an aggregate of elastically isotropic crystals. This conclusion was stated first by Lin [28].

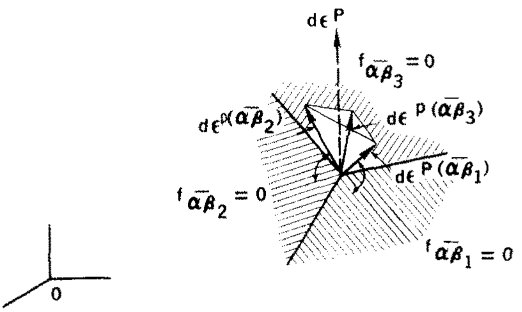

FIG. 2. Pyramid of normals to intersecting hyperplanes.

Turning attention to a unit cube of anisotropic crystals, the microstress state prior to the initiation of plastic slip can be expressed as

$$\zeta_{ij}(x) = \psi_{ijkl}(x)\sigma_{kl}. \tag{28}$$

The influence functions $\psi_{ijkl}(x)$ must be determined from an elastic solution for the nonuniform microstresses in the inhomogeneous cube, which is acted upon by the macroscopic stress state σ_{kl} applied as surface tractions. For impending slip in the $\overline{\alpha\beta}$ slip system of one crystal within the aggregate,

$$\tau_{\alpha\beta}(x) = n_{\alpha i}n_{\beta j}\zeta_{ij}(x) = n_{\alpha i}n_{\beta j}\psi_{ijkl}(x)\sigma^{0}_{kl} = \tau^{(c)0}_{\overline{\alpha\beta}}. \tag{29}$$

Thus, in six-dimensional macrostress space, the equation of the initial yield hyperplane is

$$f^0_{\overline{\alpha\beta}} = n_{\alpha i} n_{\beta j} \psi_{ijkl} \sigma^0_{kl} - \tau^{(c)0}_{\overline{\alpha\beta}} = 0. \tag{30}$$

The incremental macroscopic plastic strain, expressed in terms of the plastic slip $\mathrm{d}\gamma_{\overline{\alpha\beta}}$, is still given by equation (23). From a comparison of equations (23) and (30), we see that the macroscopic plastic strain vector and the vector gradient of $f^0_{\overline{\alpha\beta}}$ in macrostress space will not coincide unless $\psi_{ijkl} = \delta_{ik}\delta_{jl}$ (e.g., $\zeta_{ij}(x) = \sigma^0_{ij}$). This latter condition is assured if and only if the crystals are elastically isotropic. Thus, for an aggregate of anisotropic crystals, $\mathrm{d}\epsilon^p$ is not necessarily normal to the yield surface in macrostress space even though the unit cube may be macroscopically (statistically) isotropic. This observation relating to normality was also first made by Lin [28].

4. THE PRINCIPLE OF MAXIMUM PLASTIC WORK

Consider again an aggregate of elastically isotropic crystals. Let N denote the total number of slip systems throughout all the crystals of the unit cube, and let n denote the number of possible slip systems which can be activated by the macroscopic state of stress σ_{ij}. Then, from equation (18),

$$n_{\alpha i} n_{\beta j} \sigma_{ij} = \tau^{(c)}_{\overline{\alpha\beta}} - \tau^{(R)}_{\alpha\beta} \tag{31}$$

for each of the n slip systems $\overline{\alpha\beta}$. Choose another stress point σ^*_{ij} in macrostress space, lying within the elastic domain or in one of the bounding hyperplanes which constitute the yield surface of the aggregate. Then by definition,

$$n_{\alpha i} n_{\beta j} \sigma^*_{ij} \le \tau^{(c)}_{\overline{\alpha\beta}} - \tau^{(R)}_{\alpha\beta} \tag{32}$$

for all the N slip systems, and thereby for each of the n hyperplanes for which equation (31) is satisfied. Thus, subtracting equation (32) from equation (31),

$$n_{\alpha i} n_{\beta j} (\sigma_{ij} - \sigma^*_{ij}) \ge 0 \tag{33}$$

for all n. This equation is a necessary step in the proof of maximum plastic work in the aggregate. From equation (26) and the symmetry of the stress tensor σ_{ij}, the incremental macroscopic plastic work is

$$\mathrm{d}W^p \equiv \sigma_{ij}\,\mathrm{d}\epsilon^p_{ij} = \sum_k \sum_{\overline{\alpha\beta}} n_{\alpha i} n_{\beta j} \sigma_{ij}\,\mathrm{d}\gamma_{\overline{\alpha\beta}(k)}\,\Delta V'_{(k)} \tag{34}$$

where the total summation is taken over the n active slip systems. The incremental plastic work done by the stress state σ^*_{ij} on the incremental plastic strains $\mathrm{d}\epsilon^p_{ij}$ is

$$\mathrm{d}W^{p*} \equiv \sigma^*_{ij}\,\mathrm{d}\epsilon^p_{ij} = \sum_k \sum_{\overline{\alpha\beta}} n_{\alpha i} n_{\beta j} \sigma^*_{ij}\,\mathrm{d}\gamma_{\overline{\alpha\beta}(k)}\,\Delta V'_{(k)}. \tag{35}$$

Subtracting,

$$\mathrm{d}W^p - \mathrm{d}W^{p*} = \sum_k \sum_{\overline{\alpha\beta}} \{n_{\alpha i} n_{\beta j} (\sigma_{ij} - \sigma^*_{ij})\}\,\mathrm{d}\gamma_{\overline{\alpha\beta}(k)}\,\Delta V'_{(k)}. \tag{36}$$

As the inequality of equation (33) is satisfied for all n of the intersecting yield hyperplanes, and as the multipliers in equation (36) are nonnegative, we have

$$\mathrm{d}W^p - \mathrm{d}W^{p*} \equiv (\sigma_{ij} - \sigma^*_{ij})\,\mathrm{d}\epsilon^p_{ij} \ge 0 \tag{37}$$

or

$$(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\cdot d\boldsymbol{\epsilon}^p \geq 0 \tag{38}$$

which is the principle of maximum plastic work in the unit cube, expressed in terms of macroscopic stress and plastic strain.

The first statement of a principle of maximum plastic work was given by Hill [34, 35] for a rigid–plastic continuum, assuming normality as a starting point. Bishop and Hill [7] derived a principle of maximum work for a single crystal (equation 10) and for an aggregate of crystals, again neglecting the elastic portion of the strain, and established normality as a consequence (rather than a hypothesis) for a rigid–plastic polycrystalline material. Equation (38) represents an extension of the principle of maximum plastic work to an aggregate of plastic, elastically isotropic crystals.

Convexity of the aggregate yield surface in macrostress space follows from equations (33) and (21). Thus

$$(\boldsymbol{\sigma}-\boldsymbol{\sigma}^*)\cdot\nabla f_{\overline{\alpha\beta}} \geq 0 \tag{39}$$

for all intersecting hyperplanes $f_{\overline{\alpha\beta}}$ at stress point σ_{ij}. Convexity also can be concluded for an aggregate of crystals having anisotropic elastic properties from equation (30), since the elastic domain in macrostress space is the interior region bounded by the yield hyperplanes $f_{\overline{\alpha\beta}}$. Thus, equation (39) holds for anisotropic crystals as well. Equation (38), however, does not in general hold (for the model assumed herein) since the incremental plastic strain vector is not strictly normal to the aggregate yield surface for elastically anisotropic crystals. If one neglects the effects of the elastic strains, then of course both normality and Bishop and Hill's principle follow.

5. A COMPARISON WITH RESULTS FROM CONTINUUM PLASTICITY THEORY

If we rewrite equation (27) as a sum over the n possible slip systems at stress point σ_{ij}, then

$$d\boldsymbol{\epsilon}^p = \sum_n \nabla f_n(d\gamma_n\,\Delta V'_n). \tag{40}$$

Taking the step (admittedly a long one) of identifying the yield hyperplanes in this equation with the independently acting loading surfaces of the continuum plasticity theories of Koiter [36] and Sanders [37], we see that equation (40) parallels, in form, Koiter's equation

$$d\boldsymbol{\epsilon}^p = \sum_n \nabla f_n(G_n\nabla f_n\cdot d\boldsymbol{\sigma}) \tag{41}$$

for n intersecting yield surfaces (specialized to planes by Sanders), where the G_n are general scalar functions of stress and plastic strain. Since equation (41) is a generalization of the equation of small plastic strains, which is derivable for a continuously smooth (regular) yield surface f from Drucker's stability postulates [26, 38], it is of interest to determine a possible relation between equation (40) and the consequences of Drucker's postulates. These consequences are three, which can be written

$$(\sigma_{ij}-\sigma^*_{ij})\,d\epsilon^p_{ij} \geq 0, \qquad d\sigma_{ij}\,d\epsilon^p_{ij} > 0 \tag{42}$$

and

$$\mathrm{d}\epsilon_{ij}^{p} = h_{ijkl}(\sigma_{mn}, \epsilon_{mn}^{p})\,\mathrm{d}\sigma_{kl} \tag{43}$$

where σ_{ij}^{*} is any stress point lying within or on the yield surface, as before. Equation (43) is the condition of linearity, which can be deduced from Drucker's extended postulate of stability [38, 39]. This condition implies a smooth loading surface and will no longer be considered. Equations (42) imply convexity and normality, the first of which is identically the principle of maximum plastic work shown to hold for the aggregate of isotropic crystals. The second of equations (42) obviously is satisfied by equation (24) corresponding to a point of single slip. To consider the more general case of simultaneous, multiple slip (equations 25, 27, or 40), we proceed as follows. For two intersecting hyperplanes $f_{\overline{\alpha\beta}_1}$, $f_{\overline{\alpha\beta}_2}$ at macrostress point σ_{ij} (depicted graphically in Fig. 3 as a projection on the π-plane in principal macrostress space), let the loading path be such that $\mathrm{d}\boldsymbol{\sigma}\cdot\mathbf{n}_1 > 0$ and $\mathrm{d}\boldsymbol{\sigma}\cdot\mathbf{n}_2 < 0$, where $\mathbf{n}_1$ and $\mathbf{n}_2$ are unit-normals to the hyperplanes. Then the loading condition proposed

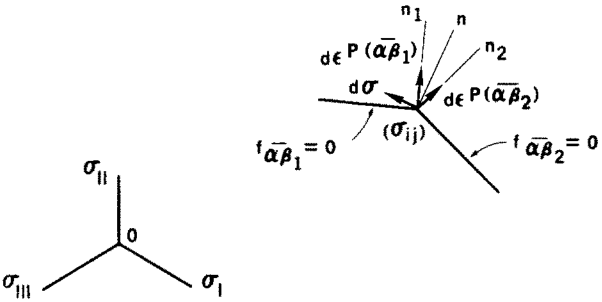

FIG. 3. Aggregate yield surface in the π-plane.

by Koiter [36], if applied to equation (40), would require that $\mathrm{d}\boldsymbol{\epsilon}^{p}$ be in the direction $\mathbf{n}_1$, while the second of equations (42) would require only that $\mathrm{d}\boldsymbol{\epsilon}^{p}$ lie in the interval bounded by $\mathbf{n}_1$ and $\mathbf{n}$, where $\mathbf{n}$ is normal to $\mathrm{d}\boldsymbol{\sigma}$ (i.e., $\mathrm{d}\boldsymbol{\sigma}\cdot\mathbf{n} = 0$). This relation was pointed out for equation (41) by Bland [40]. In the case of equation (40), if we assume incremental slip in both systems and take the scalar product of $\mathrm{d}\boldsymbol{\sigma}$ and $\mathrm{d}\boldsymbol{\epsilon}^{p(\overline{\alpha\beta}_2)}$ (using equation 23), then

$$\mathrm{d}\boldsymbol{\sigma}\cdot\mathrm{d}\boldsymbol{\epsilon}^{p(\overline{\alpha\beta}_2)} = \mathrm{d}\sigma_{ij}\,\mathrm{d}\epsilon_{ij}^{p(\overline{\alpha\beta}_2)} = n_{\alpha 2i}n_{\beta 2j}\,\mathrm{d}\sigma_{ij}\,\mathrm{d}\gamma_{\overline{\alpha\beta}_2}\,\Delta V_2' < 0. \tag{44}$$

Combining this equation with the total differential of $f_{\overline{\alpha\beta}_2}$ (from equation 18), we obtain the relation

$$\mathrm{d}f_{\overline{\alpha\beta}_2} < \mathrm{d}\tau_{\alpha\beta_2}^{(R)} - \mathrm{d}\tau_{\overline{\alpha\beta}_2}^{(c)}. \tag{45}$$

Applying Prager's condition of continuity [41] to the $\alpha\beta_2$ slip system, $\mathrm{d}f_{\overline{\alpha\beta}_2}$ must equal zero, from which it follows that

$$\mathrm{d}\tau_{\alpha\beta_2}^{(R)} > \mathrm{d}\tau_{\overline{\alpha\beta}_2}^{(c)}; \tag{46}$$

Thus, when $\mathrm{d}\boldsymbol{\sigma}\cdot\nabla f_{\overline{\alpha\beta}_2} < 0$, the increment in residual shear stress in the $\overline{\alpha\beta}_2$ slip system must exceed the increment in critical stress if slip is to occur. Taking the scalar product in the $\overline{\alpha\beta}_1$ system, we find $\mathrm{d}\tau_{\alpha\beta_1}^{(R)} < \mathrm{d}\tau_{\overline{\alpha\beta}_1}^{(c)}$. Therefore, in the case of dual slip and isotropic hardening within a single crystal volume $\Delta V'$ (such that $d\tau^{(c)}_{\overline{\alpha\beta}_1} = d\tau^{(c)}_{\overline{\alpha\beta}_2} = d\tau^{(c)}$),

$$d\tau^{(R)}_{\alpha\beta_1} < d\tau^{(R)}_{\alpha\beta_2}. \tag{47}$$

Adding the two scalar products yields

$$d\boldsymbol{\sigma} \cdot d\boldsymbol{\epsilon}^p \equiv d\sigma\, d\epsilon^p_{ij} = (d\tau^{(c)} - d\tau^{(R)}_{\alpha\beta_1})\, d\gamma_{\overline{\alpha\beta}_1} - (d\tau^{(R)}_{\alpha\beta_2} - d\tau^{(c)})\, d\gamma_{\overline{\alpha\beta}_2} \tag{48}$$

where all terms on the right side of the equation are positive. To satisfy Drucker's second condition (equation 42) would require that

$$d\gamma_{\overline{\alpha\beta}_1} > \frac{d\tau^{(R)}_{\alpha\beta_2} - d\tau^{(c)}}{d\tau^{(c)} - d\tau^{(R)}_{\alpha\beta_1}}\, d\gamma_{\overline{\alpha\beta}_2}. \tag{49}$$

Assuming there exists a range of directions of $d\boldsymbol{\sigma}$ (from $\mathbf{n} = \mathbf{n}_1$ toward $\mathbf{n} = \mathbf{n}_2$) for which $d\boldsymbol{\epsilon}^{p(\alpha\beta_2)} = 0$, then $df_{\overline{\alpha\beta}_2} \leq 0$ and $d\tau^{(R)}_{\alpha\beta_2} \leq d\tau^{(c)}$ in this interval. If $d\boldsymbol{\sigma}$ is conceived as being rotated counterclockwise from a direction that yields dual slip toward this interval corresponding to single slip, we must have $d\tau^{(R)}_{\alpha\beta_2} - d\tau^{(c)} \to 0$ as $d\boldsymbol{\epsilon}^{p(\alpha\beta_2)} \to 0$, while $d\tau^{(c)} - d\tau^{(R)}_{\alpha\beta_1}$ remains positive and nonvanishing. The right side of equation (49) then approaches zero and the inequality is satisfied. Similarly, assuming a continuously changing scalar product, the inequality is satisfied as $d\boldsymbol{\sigma}$ is rotated clockwise toward the direction $\mathbf{n} = \mathbf{n}_2$, since it is strongly satisfied there. (The righthand side is identically zero.) Thus, although this argument is necessarily limited in generality, it seems reasonable to conclude that Drucker's second condition, conceived in terms of a continuum, should apply as well to an aggregate of isotropic crystals. However, considered solely in terms of the requirement that $d\boldsymbol{\epsilon}^p$ must lie within the pyramid of normals, we see that it is a mathematically sufficient but not strictly necessary condition.

Turning our attention to an aggregate of strongly anisotropic crystals, we find that no evaluation of theoretical results—in terms of the continuum stability postulates—can be made. As can be seen from Section 3, the incremental plastic strain vector cannot be related in a simple manner to the yield hyperplanes in macrostress space, owing to the elastic inhomogeneity of the unit cube, and the normality condition does not hold. On this point it is interesting to recall a relatively recent paper by Green and Naghdi [42], wherein they develop a quite general theory of plasticity (although linear in the sense of equation 43) based upon the thermodynamic foundations and fundamental principles of continuum mechanics. For the case of isothermal, small plastic deformations, as considered here, Green and Naghdi's final equations can be stated:

$$d\epsilon^p_{ij} = \lambda\beta_{ij}\left(\frac{\partial f}{\partial\sigma_{kl}}\, d\sigma_{kl}\right) \tag{50}$$

with the additional constraint on the symmetric tensor β_{ij}

$$\beta_{kl}\left(\sigma_{kl} - \rho_0\frac{\partial U}{\partial\epsilon^p_{kl}}\right) \geq 0 \tag{51}$$

where U is the internal energy per unit mass and ρ_0 is the mass density. In Green and Naghdi's theory the tensor β_{kl} is not required to equal the gradient of the yield surface f in stress space, from which neither convexity nor normality conditions necessarily follow. Thus, there is a parallel within continuum plasticity theory for the theoretical deviation from normality predicted for an aggregate of strongly anisotropic crystals.

6. SUMMARY

Certain general theoretical results have been established for the overall macroscopic (continuum) behavior of a polycrystalline solid in which the sole mechanism of inelastic deformation is plastic slip within individual crystal grains. For an aggregate of isotropic crystals, it has been shown that the macroscopic incremental plastic strain vector is normal (or within the pyramid of normals) to the yield surface in six-dimensional macrostress space, reaffirming the conclusion of Lin [28]. In addition, the aggregate yield surface has been proved convex, from which the principle of maximum plastic work is extended to small plastic strains in an elastically isotropic polycrystalline solid. The aggregate equations are compared with equations from small strain continuum plasticity, based upon Drucker's stability postulate and the theory of independently acting loading surfaces. In the case of an aggregate of anisotropic crystals, the difficulty of deducing normality in macrostress space is discussed.

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Абстракт—Работа имеет целью определить общие характеристики зависимостей сплошной среды типа напряжение—пластическая деформация, исходя из закона пластического скольжения Шмидта в индивидуальных, кристаллических зернах. Для этого вновь подтверждаются, путем точного вывода, некоторые теоретические результаты, полученные Лином. Обобщается принцип максимальной пластической работы на случай малых, упруго-пластических деформаций в изотропной поликристаллической совокупности. Показано, что макроскопический постепенно нарастающий вектор пластической деформации по единце объема идеально зернистого тела, является совершенно нормальным к поверхности течения, в пространстве макронапряжений, только тогда, когда индивидуальные кристаллы упруго изотропны. Сравниваются результирующие уравнения для поликристаллических тел с уравнениями, вытекающими из некоторых постулятов устойчивости и термодинамических основ теории пластичности сплошной среды. Обсуждаются общие свойства подобия.